

INTERIOR REGULARITY OF FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS, II: REAL AND COMPLEX MONGE-AMPÈRE EQUATIONS

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ABSTRACT. We first consider the Dirichlet problem for the degenerate real Monge-Ampère equation:

$$(\text{Real M-A}) \quad \begin{cases} \det(u_{x^i x^j}) &= f(x) & \text{in } D \\ u &= g & \text{on } \partial D. \end{cases}$$

We prove that if $D \subset \mathbb{R}^d$ ($d \geq 2$) is a bounded strictly convex domain with C^3 boundary, $g \in C^{1,1}(\partial D)$, $0 \leq f^{1/d} \in C^{0,1}(\bar{D})$ and there exists a constant K such that $f^{1/d} + K|x|^2$ is convex in \bar{D} , then there exists a convex function $u \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ uniquely solving (Real M-A).

Then we consider the Dirichlet problem for the degenerate complex Monge-Ampère equation:

$$(\text{Complex M-A}) \quad \begin{cases} \det(u_{z^j \bar{z}^k}) &= f(z) & \text{in } D \\ u &= g & \text{on } \partial D. \end{cases}$$

We prove that if $D \subset \mathbb{C}^d$ is a bounded strictly pseudoconvex domain with C^3 boundary, $g \in C^{1,1}(\partial D)$ and $0 \leq f^{1/d} \in C^{1,1}(\bar{D})$, then there exists a plurisubharmonic function $u \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ uniquely solving (Complex M-A).

Since the second derivatives may blow up along non-tangent directions at the boundary under our regularity assumptions on the boundary data g , we also estimate the derivatives up to second order in both problems. Our technique is probabilistic by following Krylov's approach.

1. INTRODUCTION

The first problem we study is the degenerate real Monge-Ampère equation with Dirichlet boundary condition:

$$(\text{Real M-A}) \quad \begin{cases} \det(u_{x^i x^j}) &= f & \text{in } D \\ u &= g & \text{on } \partial D, \end{cases}$$

where the domain $D \subset \mathbb{R}^d$ is bounded, strictly convex and sufficiently smooth, $(u_{x^i x^j})_{d \times d}$ is the Hessian matrix of u and the function $f = f(x)$ is nonnegative. We seek a convex function uniquely solving **(Real M-A)** and investigate its regularity. This problem is very important in many fields and might be very challenging depending on the regularity and geometry of D , the regularity of f and g , as well as the positivity property of f . It has been studied extensively by many people. Particularly, (D is a bounded and strictly convex domain, unless specified.)

- In [5], Cheng and Yau proved that if D is C^2 -smooth, $g \in C^2(\bar{D})$, $f \in C_{loc}^\infty(D)$ and $0 < f \leq B \operatorname{dist}(x, \partial D)^{\beta-d-1}$, for some $B > 0$, $\beta > 0$, then $u \in C_{loc}^\infty(D)$; if $\beta = d + 1$, then $u \in C^{0,1}(\bar{D})$. They also obtained that if D is convex (not necessarily strictly convex), $g = 0$, $f \in C_{loc}^\infty(D)$ and $f > 0$, then $u \in C^0(\bar{D}) \cap C_{loc}^\infty(D)$.
- In [7], Krylov showed that if D is of C^∞ , $g \in C^\infty(\partial D)$, $f \in C^\infty(\bar{D})$ and $f > 0$, then $u \in C^\infty(\bar{D})$.
- In [3], Caffarelli, Nirenberg and Spruck obtained the same result, and they also showed in [4] that if D is $C^{3,1}$ -smooth, $g \in C^{3,1}(\partial D)$ and $f = 0$, then $u \in C^{1,1}(\bar{D})$.
- In [11], Trudinger and Urbas proved that if D is $C^{1,1}$ -smooth, $f = 0$ and $g \in C^{1,1}(\bar{D})$, then $u \in C^{0,1}(\bar{D}) \cap C_{loc}^{1,1}(D)$.
- In [9], Krylov showed that if D is $C^{3,1}$ -smooth, $g \in C^{3,1}(\partial D)$, $f^{1/d} \in C^{1,1}(\bar{D})$ and $f \geq 0$, then $u \in C^{1,1}(\bar{D})$.
- In [6], Guan, Trudinger and Wang obtained that if D is $C^{3,1}$ -smooth, $g \in C^{3,1}(\partial D)$, $f^{1/(d-1)} \in C^{1,1}(\bar{D})$ and $f \geq 0$, then $u \in C^{1,1}(\bar{D})$, which is optimal in the sense of the regularity assumption on f , due to an example by Wang in [12].

For the degenerate Monge-Ampère equation, we first note that $C^{1,1}$ -regularity is the best that we can expect, even if the boundary data g is analytic on ∂D . This can be seen by an example given in [6], by considering the unit ball in \mathbb{R}^2 as the domain D and

$$u(x_1, x_2) = [\max\{(x_1^2 - 1/2)^+, (x_2^2 - 1/2)^+\}]^2.$$

We also note that the assumption that $g \in C^{3,1}(\partial D)$ is necessary for obtaining the global $C^{1,1}$ -regularity of u . See, for example, Example 1 in [4]. While the sufficiency of $g \in C^{3,1}(\partial D)$ to obtain the global $C^{1,1}$ -regularity is established by the aforementioned papers [4, 9, 6] under various settings. Therefore, an interesting problem is investigating the interior $C^{1,1}$ -regularity of the solution to **(Real M-A)**, when g is only assumed to be in the class of $C^{1,1}(\partial D)$, whose necessity is obvious. The sufficiency for the homogeneous case is obtained by Trudinger and Urbas in the aforementioned paper [11]. Our result generalizes theirs in the sense of considering $f \geq 0$ in general, rather than $f \equiv 0$.

By thinking of the Monge-Ampère equation as a special Bellman equation with constant coefficients, we obtain regularity and solvability results on the degenerate real Monge-Ampère equation with Dirichlet boundary condition. We assume D be a bounded, C^3 -smooth and strictly convex domain and f be a function from D to $[0, \infty)$. Based on Krylov's viewpoint, we introduce the probabilistic solution to **(Real M-A)**:

$$v(x) = \inf_{\alpha \in \mathfrak{A}} E \left[g(x_{\tau^{\alpha,x}}^{\alpha,x}) - \int_0^{\tau^{\alpha,x}} d \sqrt[d]{\det((1/2)\alpha_t \alpha_t^*) f(x_t^{\alpha,x})} dt \right],$$

with

$$x_t^{\alpha, x} = x + \int_0^t \alpha_s dw_s,$$

where w_t is a Wiener process and \mathfrak{A} is the set of progressively-measurable processes α_t with values in $\mathbb{R}^{d \times d}$ satisfying $\text{tr}(\alpha_t \alpha_t^*) = 2$ for all $t \geq 0$. Our main result for **(Real M-A)** is as follows:

- If $f^{1/d} \in C^{0,1}(\bar{D})$, $g \in C^{1,1}(\partial D)$, and there exists a positive constant K such that $f^{1/d} + K|x|^2$ is convex in \bar{D} , then $v \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$, and for a.e. $x \in D$, we have the second derivative estimate

$$0 \leq v_{(\xi)(\xi)} \leq N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right), \quad \forall \xi \in \mathbb{R}^d.$$

Meanwhile, v is the unique convex solution to **(Real M-A)** in the space of $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$.

The interior $C^{1,1}$ -regularity is optimal under our regularity assumption on the boundary data g . The second derivative estimate coincides with the fact that the $C^{1,1}$ -norm of v shouldn't blow up along the tangent directions on the boundary, and says that the $C^{1,1}$ -norm of v doesn't blow up faster than $1/\text{dist}(\cdot, \partial D)^2$ in any direction near the boundary.

The second problem we study is the degenerate complex Monge-Ampère equation with Dirichlet boundary condition:

$$\textbf{(Complex M-A)} \quad \begin{cases} \det(u_{z^j \bar{z}^k}) = f & \text{in } D \\ u = g & \text{on } \partial D, \end{cases}$$

where the domain $D \subset \mathbb{C}^d$ is bounded, strictly pseudoconvex and sufficiently smooth, $(u_{z^j \bar{z}^k})_{d \times d}$ is the Hessian matrix of u and the function $f = f(z)$ is nonnegative. We seek a plurisubharmonic function uniquely solving **(Complex M-A)** and investigate its regularity.

Compared with its real counterpart, the regularity theory on the degenerate complex Monge-Ampère equation is much less developed. Many regularity results for the real Monge-Ampère equation can not be extended to the complex case, because of various reasons. Important breakthroughs have been made by the following works: (D is a bounded and strictly pseudoconvex domain.)

- In [1], Bedford and Taylor proved that if D is the unit ball, $g \in C^2(\partial D)$, $f^{1/d} \in C^2(\bar{D})$ and $f \geq 0$, then $u \in C_{loc}^{1,1}(D) \cap C(\bar{D})$. They also showed that when D is smooth but not necessarily a ball, if $f^{1/d} \in C^{0,1}(\bar{D})$ and $g \in C^2(\bar{D})$, then $u \in C^{0,1}(\bar{D})$.
- In [2], Caffarelli, Kohn, Nirenberg and Spruck showed that if D is of C^∞ , $g \in C^\infty(\partial D)$, $f \in C^\infty(\bar{D})$ and $f > 0$, then $u \in C^\infty(\bar{D})$. Regularity results under various conditions on $f = f(z, u, u_z)$ were also established there.
- In [9], Krylov obtained that if D is $C^{3,1}$ -smooth, $g \in C^{3,1}(\partial D)$, $f^{1/d} \in C^{1,1}(\bar{D})$ and $f \geq 0$, then $u \in C^{1,1}(\bar{D})$. This seems to be the

only known $C^{1,1}$ -regularity result up to boundary for the degenerate case, even when the domain D is the unit ball.

Our result generalizes Bedford and Taylor's interior $C^{1,1}$ -regularity result by allowing the domain D be any bounded, C^3 -smooth and strictly pseudoconvex domain. To be precise, we let $D \subset \mathbb{C}^d$ be a bounded, C^3 -smooth and strictly pseudoconvex domain and f be a function from D to $[0, \infty)$. Define the probabilistic solution to **(Complex M-A)** by

$$v(z) = \inf_{\alpha \in \mathfrak{A}} E \left[g(z_{\tau_{\alpha,z}}^{\alpha,z}) - \int_0^{\tau_{\alpha,z}} d \sqrt[2]{\det(\alpha_t \bar{\alpha}_t^*) f(z_t^{\alpha,z})} dt \right],$$

with

$$z_t^{\alpha,x} = z + \int_0^t \alpha_s dW_s,$$

where W_t is a normalized complex Wiener process and \mathfrak{A} be the set of progressively measurable processes α_t with values in $\mathbb{C}^{d \times d}$ satisfying $\text{tr}(\alpha_t \bar{\alpha}_t^*) = 1$ for all $t \geq 0$. Our main result for **(Complex M-A)** is the following:

- If $f^{1/d} \in C^{1,1}(\bar{D})$ and $g \in C^{1,1}(\partial D)$, then $u \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$, and for a.e. $z \in D$, we have the second derivative estimate

$$0 \leq v_{(\xi)(\xi)} \leq N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right), \quad \forall \xi \in \mathbb{R}^d.$$

Furthermore, v is the unique plurisubharmonic solution to **(Complex M-A)** in the space of $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$.

Again, the interior $C^{1,1}$ -regularity is optimal under our regularity assumption on the boundary data g . The second derivative estimate says that the $C^{1,1}$ -norm of v doesn't blow up faster than $1/\text{dist}(\cdot, \partial D)^2$ in any non-tangent direction near the boundary.

In order to keep the present paper in a reasonable length, the generalization to the Hessian equations will be worked out in a separate paper.

This paper is outlined as follows. In Section 2, we deal with the degenerate real Monge-Ampère equation by directly applying our results in [13]. In Section 3, we extend our results in Section 2 to the degenerate complex Monge-Ampère equation.

We end this section by introducing the notation. Throughout the article, the summation convention for repeated indices is assumed. Given any sufficiently smooth function $u(x)$ from \mathbb{R}^d to \mathbb{R} , for $y, z \in \mathbb{R}^d$, let

$$u_{(y)} = u_{x^i} y^i, \quad u_{(y)(z)} = u_{x^i x^j} y^i z^j, \quad u_{(y)}^2 = (u_{(y)})^2.$$

We denote the gradient vector of u by u_x and the Hessian matrix of u by u_{xx} . For any matrix σ , its tranpose is denoted by σ^* .

The notation of complex analysis will be introduced at the beginning of Section 3.

2. INTERIOR $C^{1,1}$ REGULARITY OF THE DEGENERATE REAL MONGE-AMPÈRE EQUATION

In this section, we consider the Dirichlet problem for the degenerate real Monge-Ampère equation in a strictly convex domain.

2.1. Statement of the theorem. Let D be a bounded domain in \mathbb{R}^d described by a C^3 function ψ which is non-singular on ∂D , i.e.

$$D := \{x \in \mathbb{R}^d : \psi(x) > 0\}, \quad |\psi_x| \geq 1 \text{ on } \partial D.$$

We also assume that ψ is strictly concave in \bar{D} , i.e.

$$(2.1) \quad \forall a \in \bar{S}_d^+ : \text{tr}(a) = 1, \quad \text{tr}(a\psi_{xx}) < 0 \text{ in } \bar{D},$$

where \bar{S}_d^+ denotes the set of all non-negative symmetric $d \times d$ matrices.

Let w_t be a Wiener process of dimension d , and \mathfrak{A} be the set of progressively-measurable processes α_t with values in $\mathbb{R}^{d \times d}$ satisfying $\text{tr}(\alpha_t \alpha_t^*) = 2, \forall t \geq 0$. Introduce a family of controlled diffusion processes

$$x_t^{\alpha, x} = x + \int_0^t \alpha_s dw_s, \quad \forall \alpha_t \in \mathfrak{A}.$$

Denote $\tau^{\alpha, x}$ the first exit time of $x_t^{\alpha, x}$ from D .

Let f and g be bounded measurable functions on \bar{D} with values in $[0, \infty)$ and \mathbb{R} , respectively.

Theorem 2.1. *Let*

$$(2.2) \quad v(x) = \sup_{\alpha \in \mathfrak{A}} E \left[g(x_{\tau^{\alpha, x}}^{\alpha, x}) + \int_0^{\tau^{\alpha, x}} \sqrt[d]{\det(a_t^\alpha)} f(x_t^{\alpha, x}) dt \right],$$

with

$$a_t^\alpha = \frac{1}{2} \alpha_t \bar{\alpha}_t^*.$$

If $f, g \in C^{0,1}(\bar{D})$, then $v \in C_{loc}^{0,1}(D) \cap C(\bar{D})$, and for a.e. $x \in D$,

$$(2.3) \quad |v(\xi)| \leq N \left(|\xi| + \frac{|\psi(\xi)|}{\psi^{1/2}} \right), \quad \forall \xi \in \mathbb{R}^d,$$

where the constant $N = N(|f|_{0,1,D}, |g|_{0,1,D}, |\psi|_{3,D}, d)$.

If $f \in C^{0,1}(\bar{D})$, $g \in C^{1,1}(\bar{D})$ and there exists a constant K such that $f + K|x|^2$ is convex in \bar{D} , then $v \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$, and for a.e. $x \in D$,

$$(2.4) \quad -N \left(|\xi|^2 + \frac{\psi^2(\xi)}{\psi} \right) \leq v_{(\xi)(\xi)} \leq 0, \quad \forall \xi \in \mathbb{R}^d,$$

where the constant $N = N(|f|_{0,1,D}, |g|_{1,1,D}, |\psi|_{3,D}, K, d)$. Meanwhile, $u = -v$ is the unique convex solution in $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ of the Dirichlet problem for the degenerate Monge-Ampère equation:

$$(2.5) \quad \begin{cases} \det(u_{xx}) &= d^{-d} f^d & \text{a.e. in } D \\ u &= -g & \text{on } \partial D, \end{cases}$$

satisfying the second derivative estimate: for a.e. $x \in D$,

$$0 \leq u_{(\xi)(\xi)} \leq N \left(|\xi|^2 + \frac{\psi_{(\xi)}^2}{\psi} \right), \quad \forall \xi \in \mathbb{R}^d.$$

Remark 2.1. The assumption that $f + K|x|^2$ is convex in \bar{D} for some constant K is weaker than the assumption that $f \in C^{1,1}(\bar{D})$.

Remark 2.2. The first derivative estimate (2.3) is not new. It has been obtained in [10], in which an example (see Remark 5.1 in [10]) was also given showing that (2.3) is sharp. The interior $C^{1,1}$ -regularity result and the second derivative estimate (2.4) are new. However, the author doesn't know whether the estimate is sharp.

2.2. Proof of Theorem 2.1. We first show that Theorems 2.1, 2.2 and 2.3 in [13] are applicable for $v(x)$ given in (2.2) by verifying Assumptions 2.1 and 2.2 in [13] and the weak non-degeneracy condition (See Remark 2.1 in [13]). Then we show that the associated dynamic programming equation of $v(x)$ is equivalent to the Monge-Ampère equation in (2.5).

Proof. We apply Theorems 2.1, 2.2 and 2.3 in [13] with

$$A = \{\alpha \in \mathbb{R}^{d \times d} : \text{tr}(\alpha \alpha^*) = 2\},$$

$$\sigma^\alpha = \alpha, \quad a^\alpha = \frac{1}{2} \alpha \alpha^*, \quad b^\alpha = 0, \quad c^\alpha = 0, \quad f^\alpha = \sqrt[d]{\det(a^\alpha)} f.$$

In this situation,

$$\mathbb{A} = \{a^\alpha : \alpha \in A\} = \{a \in \bar{\mathcal{S}}_d^+ : \text{tr}(a) = 1\}.$$

It follows that (2.1) reads

$$\sup_{\alpha \in A} \text{tr}(a^\alpha \psi) < 0.$$

Due to the compactness of \mathbb{A} , by replacing ψ with $N\psi$, we may assume

$$\sup_{\alpha \in A} \text{tr}(a^\alpha \psi) \leq -1,$$

which is exactly Assumption 2.1 in [13].

Next, we notice that for any orthogonal matrix O of size $d \times d$,

$$\text{tr}(O a^\alpha O^*) = \text{tr}(O^* O a^\alpha) = \text{tr}(a^\alpha) = 1,$$

which implies that

$$O \mathbb{A} O^* \subset \mathbb{A} = O^*(O \mathbb{A} O^*) O \subset O \mathbb{A} O^*.$$

Therefore Assumption 2.1 in [13] also holds.

To verify the weakly non-degeneracy condition we consider $a^{\alpha_0} = (1/d)I_{d \times d} \in \mathbb{A}$, then by (2.5) and (2.6) in [13] we have

$$\mu = \inf_{|\zeta|=1} \sup_{\alpha \in A} (a^\alpha)_{ij} \zeta^i \zeta^j \geq \inf_{|\zeta|=1} (a^{\alpha_0})_{ij} \zeta^i \zeta^j = 1/d > 0.$$

Therefore Theorems 2.1, 2.2 and 2.3 in [13] are true for $v(x)$ defined by (2.2).

It remains to prove the second inequality in (2.4) and the equivalence between the associated dynamic programming equation

$$(2.6) \quad \sup_{\alpha \in A} \left[(a^\alpha)_{ij} v_{x^i x^j} + \sqrt[d]{\det(a^\alpha)} f \right] = 0$$

and the Monge-Ampère equation in (2.5). They are well-known facts, which were actually Lemma 2 in Section 3.2 of [8]. For the sake of completeness and the convenience of the reader we give the following argument.

First, we rewrite (2.6) as

$$(2.7) \quad \sup_{a \in \mathbb{A}} \left[\operatorname{tr}(a v_{xx}) + \sqrt[d]{\det(a)} f \right] = 0.$$

In particular, in D ,

$$(2.8) \quad \operatorname{tr}(a v_{xx}) + \sqrt[d]{\det(a)} f \leq 0$$

for each $a \in \mathbb{A}$. For any fixed $\zeta \in \mathbb{R}^d$ with $|\zeta| = 1$ and $a = \zeta \zeta^*$, from (2.8) we get

$$(v_{xx} \zeta, \zeta) \leq -\sqrt[d]{\det(a)} f \leq 0,$$

which proves the second inequality in (2.4), and means that v is a concave function in D .

Next, take $\delta > 0$ and set

$$a = (\delta I - v_{xx})^{-1} c_\delta, \quad c_\delta^{-1} = \operatorname{tr} [(\delta I - v_{xx})^{-1}].$$

Then $\operatorname{tr}(a) = 1$ and (2.8) yields

$$\operatorname{tr} [(\delta I - v_{xx})^{-1} v_{xx}] + [\det(\delta I - v_{xx})]^{-1/d} f \leq 0.$$

It follows that

$$\begin{aligned} f &\leq [\det(\delta I - v_{xx})]^{1/d} \operatorname{tr} [(\delta I - v_{xx})^{-1} (-v_{xx})] \\ &= [\det(\delta I - v_{xx})]^{1/d} \{ \operatorname{tr}(I) - \delta \operatorname{tr} [(\delta I - v_{xx})^{-1}] \} \\ &\leq [\det(\delta I - v_{xx})]^{1/d} d. \end{aligned}$$

Therefore we have

$$d^{-d} f^d \leq \det(\delta I - v_{xx}).$$

By letting $\delta \downarrow 0$ we obtain

$$d^{-d} f^d \leq \det(-v_{xx}) = \det(u_{xx}).$$

In fact, here an equality holds instead of the inequality. To prove this suppose that at some point $x_0 \in D$, we have

$$d^{-d} f^d < \det(u_{xx}).$$

Then, in particular, $\det(u_{xx}) > 0$ and u_{xx} is non-degenerate at x_0 . Take a matrix $a_0 \in \mathbb{A}$ which attains the supremum in (2.7) at x_0 . Since

$$-\operatorname{tr}(a_0 v_{xx}) = \operatorname{tr}(a_0 u_{xx}) \geq \lambda_{\min} \operatorname{tr}(a_0) > 0,$$

where λ_{\min} is the smallest eigenvalue of the strictly positive matrix u_{xx} , we see that

$$\det(a_0) > 0.$$

Now by the fact that the geometric mean is not bigger than the arithmetic mean,

$$\begin{aligned} f[\det(a_0)]^{1/d} &= -\operatorname{tr}(a_0 v_{xx}) = \operatorname{tr}(a_0 u_{xx}) = \operatorname{tr}(\sqrt{a_0} u_{xx} \sqrt{a_0}) \\ &\geq d[\det(\sqrt{a_0} u_{xx} \sqrt{a_0})]^{1/d} = d[\det(u_{xx})]^{1/d} [\det(a_0)]^{1/d}. \end{aligned}$$

It follows that at x_0 ,

$$d^{-d} f^d \geq \det(u_{xx}),$$

which gives the desired contradiction. Hence the equivalence of (2.6) and the Monge-Ampère equation in (2.5) is proved. \square

3. INTERIOR $C^{1,1}$ REGULARITY OF THE DEGENERATE COMPLEX MONGE-AMPÈRE EQUATION

In this section, we consider the Dirichlet problem for degenerate complex Monge-Ampère equation in a strictly pseudoconvex domain.

3.1. Statement of the theorem. We use the following standard notation: \mathbb{C} denotes the set of all complex numbers; and

$$\begin{aligned} z &= (z^1, \dots, z^d) = (x^1 + ix^{d+1}, \dots, x^k + ix^{d+k}, \dots, x^d + ix^{d+d}) \\ &= (x^1, \dots, x^d) + i(x^{d+1}, \dots, x^{d+d}) =: \operatorname{Re} z + i \operatorname{Im} z \end{aligned}$$

is an element of \mathbb{C}^d . We also use the following notation of partial differential operators:

$$\begin{aligned} u_{z^k} &= \frac{1}{2}(u_{x^k} - iu_{y^k}), & u_{\bar{z}^k} &= \frac{1}{2}(u_{x^k} + iu_{y^k}) \\ u_{z^k \bar{z}^j} &= (u_{z^k})_{\bar{z}^j} & u_{z \bar{z}} &= (u_{z^k \bar{z}^j})_{1 \leq j, k \leq d} \end{aligned}$$

Moreover, for any $\xi, \eta \in \mathbb{C}^d$, we define

$$u_{(\xi)} = u_{z^k} \xi^k + u_{\bar{z}^k} \bar{\xi}^k, \quad u_{(\xi)(\eta)} = (u_{(\xi)})_{(\eta)}$$

Since any function u from \mathbb{C}^d to \mathbb{R} can be viewed as a function from \mathbb{R}^{2d} to \mathbb{R} , by abuse of notation we write $u(z) = u(x)$ with $x = (\operatorname{Re} z, \operatorname{Im} z)$. As a result, we see that

$$u_{(\xi)}(z) = u_{(\operatorname{Re} \xi, \operatorname{Im} \xi)}(x), \quad u_{(\xi)(\eta)}(z) = u_{(\operatorname{Re} \xi, \operatorname{Im} \xi)(\operatorname{Re} \eta, \operatorname{Im} \eta)}(x)$$

Let D be a bounded domain in \mathbb{C}^d described by a C^3 function ψ which is non-singular on ∂D , i.e.

$$D := \{z \in \mathbb{C}^d : \psi(z) > 0\}, \quad |\psi_z| \geq 1 \text{ on } \partial D.$$

We also assume that ψ is strictly plurisuperharmonic in \bar{D} , i.e.

$$(3.1) \quad \forall a \in \bar{\mathcal{H}}_d^+ : \operatorname{tr}(a) = 1, \quad \operatorname{tr}(a\psi_{z\bar{z}}) < 0 \text{ in } \bar{D},$$

where $\bar{\mathcal{H}}_d^+$ denotes the set of all non-negative Hermitian $d \times d$ matrices.

Let W_t be a normalized complex Wiener process of dimension d , i.e. a d -dimensional stochastic process $W_t = (W_t^1, \dots, W_t^d)$ with values in \mathbb{C}^d given by

$$W_t^j = \frac{1}{\sqrt{2}}(w_t^{j,1} + iw_t^{j,2}), \quad t \geq 0, \quad 1 \leq j \leq d,$$

where the processes $(w_t^{j,1}, w_t^{j,2})_{1 \leq j \leq d}$ are independent real Wiener processes.

Let \mathfrak{A} be the set of progressively-measurable processes α_t with values in $\mathbb{C}^{d \times d}$ satisfying $\text{tr}(\alpha_t \bar{\alpha}_t^*) = 1, \forall t \geq 0$, and

$$A = \{\alpha \in \mathbb{C}^{d \times d} : \text{tr}(\alpha \bar{\alpha}^*) = 1\}.$$

If we denote $\alpha \bar{\alpha}^*$ by a^α , then

$$\mathbb{A} := \{a^\alpha : \alpha \in A\} = \{a \in \bar{\mathcal{H}}_d^+ : \text{tr } a = 1\}.$$

Introduce a family of controlled complex diffusion processes

$$z_t^{\alpha, x} = z + \int_0^t \alpha_s dW_s, \quad \forall \alpha_t \in \mathfrak{A}.$$

Denote $\tau^{\alpha, z}$ the first exit time of $z_t^{\alpha, z}$ from D .

Let f and g be bounded measurable functions on \bar{D} with values in $[0, \infty)$ and \mathbb{R} respectively.

Theorem 3.1. *Let*

$$(3.2) \quad v(z) = \sup_{\alpha \in \mathfrak{A}} E \left[g(z_{\tau^{\alpha, z}}^{\alpha, z}) + \int_0^{\tau^{\alpha, z}} \sqrt[d]{\det(a_t^\alpha)} f(z_t^{\alpha, z}) dt \right],$$

with

$$a_t^\alpha = \alpha_t \bar{\alpha}_t^*.$$

If $f, g \in C^{0,1}(\bar{D})$, then $v \in C_{loc}^{0,1}(D) \cap C(\bar{D})$, and for a.e. $x \in D$,

$$(3.3) \quad |v(\xi)| \leq N \left(|\xi| + \frac{|\psi(\xi)|}{\psi^{1/2}} \right), \quad \forall \xi \in \mathbb{C}^d,$$

where the constant $N = N(|f|_{0,1,D}, |g|_{0,1,D}, |\psi|_{3,D}, d)$.

If $f, g \in C^{1,1}(\bar{D})$, then $v \in C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$, and for a.e. $x \in D$,

$$(3.4) \quad -N \left(|\xi|^2 + \frac{\psi^2(\xi)}{\psi} \right) \leq v_{(\xi)}(\xi) \leq 0, \quad \forall \xi \in \mathbb{C}^d,$$

where the constant $N = N(|f|_{1,1,D}, |g|_{1,1,D}, |\psi|_{3,D}, d)$. Meanwhile, $u = -v$ is the unique plurisubharmonic solution in $C_{loc}^{1,1}(D) \cap C^{0,1}(\bar{D})$ of the Dirichlet problem for the degenerate complex Monge-Ampère equation:

$$(3.5) \quad \begin{cases} \det(u_{z\bar{z}}) &= d^{-d} f^d & \text{a.e. in } D \\ u &= -g & \text{on } \partial D, \end{cases}$$

satisfying the second derivative estimate: for a.e. $x \in D$,

$$0 \leq u_{(\xi)}(\xi) \leq N \left(|\xi|^2 + \frac{\psi^2(\xi)}{\psi} \right), \quad \forall \xi \in \mathbb{C}^d.$$

Remark 3.1. When studying the $C^{1,1}$ -regularity of v , the regularity assumption on f we need is actually weaker than $C^{1,1}(\bar{D})$. Similarly to the real case, we only need that the generalized second derivatives of f are bounded from below. Therefore, the assumption that $f \in C^{1,1}(\bar{D})$ can be replaced with the assumptions that $f \in C^{0,1}(\bar{D})$ and $f + K|x|^2$ is convex for some constant K , where we treat $f = f(x)$ as a function of $2d$ real variables with $x = (\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^{2d}$. When f is sufficiently smooth, the condition that $f + K|x|^2$ is convex is equivalent to

$$\operatorname{Re} \left(f_{z^j \bar{z}^k}(z) \xi^j \bar{\xi}^k \right) + f_{z^j \bar{z}^k}(z) \xi^j \bar{\xi}^k \geq -K|\xi|^2, \quad \forall \xi \in \mathbb{C}^d.$$

3.2. Proof of Theorem 3.1. We prove Theorem 3.1 by making use of Theorems 2.1, 2.2 and 2.3 in [13].

Proof. We first define the following homomorphisms:

$$\Phi : \mathbb{C}^d \rightarrow \mathbb{R}^{2d}; z \mapsto \begin{pmatrix} \operatorname{Re} z \\ \operatorname{Im} z \end{pmatrix}$$

and

$$\Phi : \mathbb{C}^{d \times d} \rightarrow \mathbb{R}^{2d \times 2d}; \alpha \mapsto \begin{pmatrix} \operatorname{Re} \alpha & \operatorname{Im} \alpha \\ -\operatorname{Im} \alpha & \operatorname{Re} \alpha \end{pmatrix}.$$

To rewrite the value function in (3.2) as a function on $\mathbb{R}^{2d \times 2d}$, we notice that

$$\Phi(z_t^{\alpha, z}) = \Phi z + \int_0^t \frac{1}{\sqrt{2}} (\Phi \alpha_s) dw_s$$

where w_t is a Wiener process of dimension $2d$.

For any Hermitian matrix a , there is a unitary matrix U such that $Ua\bar{U}^*$ is a real diagonal matrix M . We have,

$$\det(a) = \det(\bar{U}^* M U) = \det(M).$$

If U is a unitary matrix, then ΦU is an orthogonal matrix. Moreover

$$\begin{aligned} \det(\Phi a) &= \det[\Phi(\bar{U}^* M U)] = \det[\Phi(\bar{U}^*) \Phi M \Phi U] \\ &= \det[(\Phi U)^* \Phi M \Phi U] = \det(\Phi M) = [\det(M)]^2. \end{aligned}$$

Therefore

$$\sqrt[d]{\det(a_t^\alpha)} = \sqrt[2d]{\det \Phi(\alpha_t \bar{\alpha}_t^*)} = \sqrt[2d]{\det(\Phi \alpha_t \Phi \bar{\alpha}_t^*)} = \sqrt[2d]{\det[(\Phi \alpha_t)(\Phi \alpha_t)^*]}.$$

If we use the notation:

$$\beta = \Phi \alpha, \quad \mathfrak{B} = \Phi \mathfrak{A} = \{\Phi \alpha : \alpha \in \mathfrak{A}\},$$

then we can rewrite (3.2) as

$$(3.6) \quad v(x) = \sup_{\beta \in \mathfrak{B}} E_x^\beta \left[g(x_\tau) + \int_0^\tau \sqrt[2d]{\det(\beta_t \beta_t^*)} f(x_t) dt \right],$$

where

$$x_t^{\beta, x} = x + \int_0^t \frac{1}{\sqrt{2}} \beta_s dw_s.$$

By noticing that

$$\mathrm{tr}[(\Phi a)\psi_{xx}] = 4 \mathrm{tr}(a\psi_{z\bar{z}}),$$

Assumption 2.1 in [13] is satisfied by replacing ψ with $N\psi$ for sufficiently constant N . However, if we have a try on applying Theorems 2.1-2.3 in [13] directly, we should fail at Assupmtion 2.2 in [13]. Because Assupmtion 2.2 in [13] doesn't hold for

$$\mathbb{B} = \Phi\mathbb{A} = \{\Phi(a^\alpha) : a^\alpha \in \mathbb{A}\}.$$

More precisely, since

$$\mathbb{B} = \left\{ \begin{pmatrix} S & T \\ -T & S \end{pmatrix} : S \in \bar{\mathcal{S}}_d^+, T \text{ is skew symmetric, } \mathrm{tr}(S) = 1 \right\},$$

the relation

$$O\mathbb{B}O^* = \mathbb{B}$$

doesn't hold for all orthogonal matrix O of size $2d \times 2d$.

Fortunately, for any unitary matrix of size $d \times d$, we have

$$U\mathbb{A}\bar{U}^* = \mathbb{A}.$$

which can play a role of Assumption 2.2 in [13]. Indeed, we observe that

$$\mathbb{B} = \Phi\mathbb{A} = \Phi(U\mathbb{A}\bar{U}^*) = \Phi U\Phi\mathbb{A}\Phi(\bar{U}^*) = (\Phi U)\mathbb{B}(\Phi U)^*.$$

Moreover, we note that if $Q \in \mathbb{C}^{d \times d}$ is skew Hermitian, then e^Q is unitary, and therefore $e^{\Phi Q} = \Phi(e^Q)$ satisfies

$$(e^{\Phi Q})\mathbb{B}(e^{\Phi Q})^* = \mathbb{B}$$

Therefore, in order to apply Theorems 2.1-2.3 in [13], it suffice to find a suitable matrix function $P = P(x, \xi)$ from $D \times \mathbb{R}^{2d}$ to $\mathbb{R}^{2d \times 2d}$, such that P can be expressed as ΦQ , where Q is skew Hermitian, and $P(x, \xi)$ satisfies all properties it has in the proof of Lemma 7.1 in [13].

To construct P let us start from looking at the equation (7.1) in [13], the most crucial property it satisfies in the proof of Lemma 7.1 in [13]. We define

$$\chi : D_\delta^\lambda \times \mathbb{C}^d \rightarrow \mathbb{C}; (z, \xi) \mapsto -\frac{\psi_{\bar{z}k}(\psi_{zk})_{(\xi)}}{|\psi_{\bar{z}}|^2}$$

and

$$R : D_\delta^\lambda \times \mathbb{C}^d \rightarrow \mathbb{C}^{d \times d}; (z, \xi) \mapsto \left(\frac{(\psi_{zk})_{(\xi)}\psi_{\bar{z}j} - (\psi_{\bar{z}j})_{(\xi)}\psi_{zk}}{|\psi_{\bar{z}}|^2} \right)_{1 \leq j, k \leq d},$$

which are analogous to ρ and P in Lemma 7.1 in [13].

We claim that

$$(3.7) \quad (\psi_{\bar{z}})_{(\xi)} + R\psi_{\bar{z}} + \chi\psi_{\bar{z}} = 0.$$

Indeed, we have

$$\begin{aligned}
& \left[(\psi_{\bar{z}})_{(\xi)} + R\psi_{\bar{z}} + \chi\psi_{\bar{z}} \right]^j \\
&= (\psi_{\bar{z}j})_{(\xi)} + \frac{(\psi_{z^k})_{(\xi)}\psi_{\bar{z}j} - (\psi_{\bar{z}j})_{(\xi)}\psi_{z^k}}{\psi_z\psi_{\bar{z}}} \psi_{\bar{z}^k} - \frac{\psi_{\bar{z}^k}(\psi_{z^k})_{(\xi)}\psi_{\bar{z}j}}{\psi_z\psi_{\bar{z}}} \\
&= (\psi_{\bar{z}j})_{(\xi)} \left[1 - \frac{\psi_{z^k}\psi_{\bar{z}^k}}{\psi_z\psi_{\bar{z}}} \right] = 0.
\end{aligned}$$

Next, we notice that χ is not real in general, so we decompose it as $\chi = \rho + i\kappa$, where ρ and κ are real valued. If we denote $R + i\kappa I$ as Q , the equation (3.7) can be rewritten as

$$(3.8) \quad (\psi_{\bar{z}})_{(\xi)} + Q\psi_{\bar{z}} + \rho\psi_{\bar{z}} = 0.$$

We emphasize that ρ is real and Q is skew Hermitian. From (3.8) and the fact that $\psi_x = 2\Phi(\psi_{\bar{z}})$, we obtain

$$\begin{aligned}
0 &= \left(\alpha\alpha^* ((\psi_{\bar{z}})_{(\xi)} + Q\psi_{\bar{z}} + \rho\psi_{\bar{z}}), (\psi_{\bar{z}})_{(\xi)} + Q\psi_{\bar{z}} + \rho\psi_{\bar{z}} \right) \\
&= \left(\Phi \left(\alpha\alpha^* ((\psi_{\bar{z}})_{(\xi)} + Q\psi_{\bar{z}} + \rho\psi_{\bar{z}}) \right), \Phi \left((\psi_{\bar{z}})_{(\xi)} + Q\psi_{\bar{z}} + \rho\psi_{\bar{z}} \right) \right) \\
&= \frac{1}{4} \left(\beta\beta^* ((\psi_x)_{(\xi)} + (\Phi Q)\psi_x + \rho\psi_x), (\psi_x)_{(\xi)} + (\Phi Q)\psi_x + \rho\psi_x \right)
\end{aligned}$$

Therefore if we let $P = \Phi Q$,

$$\psi_{(\xi)(\beta^k)} + \rho\psi_{(\beta^k)} + \psi_{(P\beta^k)} = 0.$$

It is also not hard to see that P and ρ we define here satisfy all the other property in Lemma 7.1 in [13]. As a result, we can apply Theorem 2.1-2.3 in [13] to obtain all regularity results of v defined by (3.2) stated in Theorem 3.1.

It remains to verify that the associated dynamic programming equation is equivalent to the complex Monge-Ampère equation in (3.5). To write down the real Bellman equation we note that its diffusion term is $(1/\sqrt{2})\beta$. Hence the associated dynamic programming equation is the real Bellman equation

$$(3.9) \quad \sup_{\beta \in B} \left[(1/4) \operatorname{tr}(\beta\beta^* v_{xx}) + \sqrt[2d]{\det(\beta\beta^*)} f \right] = 0,$$

which is equivalent to

$$(3.10) \quad \sup_{a \in A} \left\{ \operatorname{tr}[(\Phi a)v_{xx}] + 4 \sqrt[2d]{\det(\Phi a)} f \right\} = 0.$$

To write down the corresponding complex Bellman equation, it suffices to notice that

$$\operatorname{tr}[(\Phi a)v_{xx}] = 4 \operatorname{tr}(av_{z\bar{z}}).$$

Therefore the complex Bellman equation is

$$(3.11) \quad \sup_{a \in A} \left[\operatorname{tr}(av_{z\bar{z}}) + \sqrt[d]{\det(a)} f \right] = 0,$$

which has the same form of (2.6). Therefore, the equivalence between (3.11) and the complex Monge-Ampère equation in (3.5) can be verified by repeating the argument right after (2.7). The proof is complete. \square

4. ACKNOWLEDGEMENTS

The author wishes to express sincere gratitude towards his PhD advisor, Professor Nicolai V. Krylov, for illuminating suggestions and the financial support during the preparation of this paper. The author is also very grateful to Professor Hongjie Dong for inspiring discussions on fully nonlinear PDE theory when the author visited Brown University.

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